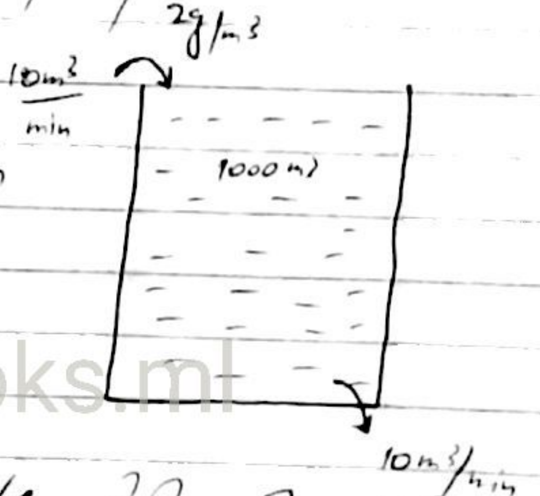
(Lec 9,10) Modeling of differential equations:- 22/6/15

Q no 1:-

Water flows into pool at $10 m^3/min$ and out the other end with same rate. The pool volume is $1000 m^3$. Initially pool has 100 gm of pollutants and there is additional pollutants flowing into the pool at rate of $2 g/m^3$. What is formula for no. of grams of pollutants into pool as a function of time



Sol:-

$x_{(t)}$ = Pollutants in grams at time t

$$\frac{dx}{dt} = R_{in} - R_{out}$$

R_{in} = Inward Rate = (Liquid flow in)(Concentration of pollutants in)

$$= \left(10\ m^3/min\right)\left(2\ g/m^3\right) = 20\ g/min$$

R_{out} = Outward Rate = (Liquid flow out)(Concentration of pollutants out)

$$= \left(\frac{10\ m^3}{min}\right)\left(\frac{x_{(t)}}{1000}\right) = \left(x/100\right) gm/min$$

$$\frac{dx}{dt} = R_{in} - R_{out} = 20 - \frac{x}{100} = \boxed{\frac{2000 - x}{100}}$$

It is seperable,

$$\frac{dx}{2000-x} = dt/100$$

Integrate it,

$$\int \frac{dx}{2000-x} = \frac{1}{100}\int 1\,dt$$

Suppose

$2000 - x = u$

$-dx = du$

$dx = (-du)$

by substitution,

$$\int -\frac{du}{u} = \frac{1}{100}t + c$$

$$-\ln u = \frac{t}{100} + c$$

$$\ln u = -\frac{t}{100} + c$$

$$u = e^{(-t/100 + c)}$$

$$2000 - x = e^{-t/100} \cdot c$$

$$\boxed{x_{(t)} = 2000 - ce^{-t/100}}$$ General Solution

Initial Condition,

$t = 0 \qquad x_{(0)} = 100$ gm

$100 = 2000 - ce^{0}$

$100 = 2000 - c$

$c = 2000 - 100$

$c = 1900$ put it in general solution

$$\boxed{X_{(t)} = 2000 - 1900\, e^{-t/100}}$$ particular Solution

Checks:-

i) When $t = 0$

$X_{(0)} = 2000 - 1900\, e^{-t/100}$

$= 100$ gm

ii) When $t = \infty$

$X_{(\infty)} = 2000 - 1900\, e^{-\infty}$

$= 2000 - 1900 \cdot \frac{1}{e^{\infty}}$

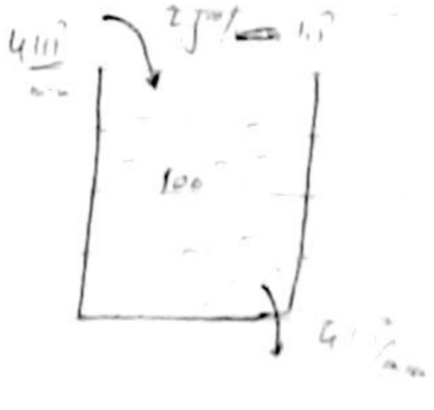
$X_{(\infty)} = 2000$ gm

Ques no 2-

A tank contains 100 litre of water in which 2gm of salt is dissolved. Water containing 2gm/litre of salt is poured into tank @ 4 lit/min and well mixed solution flows out at same rate from same. How much salt is in tank after t minutes.

Sol-

$\frac{dx}{dt} = R_{in} - R_{out}$



$$\frac{dx}{dt} = \left(4\ lit/_{min}\right)\left(2\ gm/_{lit}\right) - \left(4\ lit/_{min}\right)\left(\frac{x}{100}\ gm/_{lit}\right)$$

$$= 8 - \frac{4x}{100} = \frac{800 - 4x}{100}$$

It is separable,

$$\frac{dx}{800-4x} = \frac{dt}{100}$$

Integrate it,

$$\frac{1}{4}\int \frac{dx}{200-x} = \frac{1}{100}\int 1\,dt$$

Suppose

$200 - x = u$

$-dx = du$

$dx = (-du)$

$$\int -\frac{du}{u} = \frac{4t}{100} + c$$

$$-\ln u = \frac{4t}{100} + c$$

$$u = e^{(-4t/100 + c)}$$

$$200 - x = e^{-4t/100} \cdot c$$

$$\boxed{x_{(t)} = 200 - ce^{-4t/100}}$$ General Solution.

Initial Condition,

$t = 0 \qquad X_{(0)} = 20\ gm$

$20 = 200 - ce^{0}$

$c = 200 - 20 = 180$

$$\boxed{X_{(t)} = 200 - 180\, e^{-4t/100}}$$ Particular Solution.

Check:-

i) at $t = 0$

$X_{(0)} = 200 - 180\, e^{-0/100}$

$= 200 - 180$

$= 20\ gm$

ii) at $t = \infty$

$X_{(\infty)} = 200 - 180\, e^{-\infty}$

$= 200 - 180 \cdot \frac{1}{e^{\infty}}$

$X_{(\infty)} = 200\ gm$